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VARIANTS OF THE MAXIMAL DOUBLE HILBERT TRANSFORM

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ABSTRACT. We prove the boundedness on $L_p(T^2)$, $1 < p < \infty$, of two variants of the double Hilbert transform and maximal double Hilbert transform. They have an application to a problem of almost everywhere convergence of double Fourier series.

Introduction. In this paper we study two variants of the double Hilbert transform

$$Df(x,y) = \iint_{|y'|\leq\pi, |x'|\leq\pi} 1/x'y' f(x-x', y-y')\, dx'\, dy'$$

and maximal double Hilbert transform

$$\tilde{D}f(x,y) = \operatorname*{Sup}_{\varepsilon,\delta>0} \left| \iint_{\delta<|y'|\leq\pi, \varepsilon<|x'|\leq\pi} 1/x'y' f(x-x', y-y')\, dx'\, dy' \right|$$

which are, roughly speaking, of the following kind. First we consider

$$H_1 f(x,y) = \iint_{(x',y')\in A} 1/x'y' f(x-x', y-y')\, dx'\, dy'$$

and

$$\tilde{H}_1 f(x,y) = \operatorname*{Sup}_{\varepsilon,\delta>0} \left| \iint_{\substack{(x',y')\in A \\ |x'|>\varepsilon, |y'|>\delta}} 1/x'y' f(x-x', y-y')\, dx'\, dy' \right|,$$

where $A \subset \{(x',y')\colon |x'| \leq \pi,\ |y'| \leq \pi\} = T^2$ is a fixed region symmetrical with respect to the axes x' and y' but, except for this natural requirement, quite general. (The cut-off of the kernel $1/x'y'$, given by $\chi_A(x',y')$, is actually smoothly done. See §2 for the exact definition.) Secondly, we consider

$$H_2 f(x,y) = \iint_{(x',y')\in A_x} 1/x'y' f(x-x', y-y')\, dx'\, dy'$$

and

$$\tilde{H}_2 f(x,y) = \operatorname*{Sup}_{\delta>0} \left| \iint_{\substack{(x',y')\in A_x \\ |y'|>\delta}} 1/x'y' f(x-x', y-y')\, dx'\, dy' \right|,$$

where, for every x, the domain of integration A_x is symmetrical with respect to the axes and otherwise is quite general.

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We shall prove that $H_1, \tilde{H}_1, H_2, \tilde{H}_2$ are bounded operators from $L_p(T^2)$ to itself, $1 < p < \infty$. Moreover, we shall give a pointwise estimate from above of $\tilde{H}_1$ and $\tilde{H}_2$ similar to the known one concerning $\tilde{D}$ (see [**6**, p. 218]). Namely, we are going to prove that

$$\begin{aligned}\tilde{H}_1 f(x,y) \le c\{&M_{x'}M_{y'}f(x,y) + M_{x'}\tilde{H}_{y'}f(x,y)\\ &+ M_{y'}\tilde{H}_{x'}f(x,y) + M_{x'}M_{y'}(H_1 f)(x,y)\},\end{aligned} \tag{1}$$

$$\tilde{H}_2 f(x,y) \le c\{M_{y'}\tilde{H}_{x'}f(x,y) + M_{\overline{y}}(H_2 f(x,\overline{y}))(y)\} \tag{2}$$

where $M_{x'}$ is the Hardy-Littlewood maximal function acting on the x' variable, $\tilde{H}$ denotes a variant of the maximal Hilbert transform (see §1). These results apply to a problem of almost everywhere convergence of double Fourier series [**5**], where it appears that $\tilde{H}_1$ and $\tilde{H}_2$ play the same central role that the maximal Hilbert transform plays in the proof of a.e. convergence of Fourier series of one variable [**1, 2, 4**].

Let us observe that H_1 and $\tilde{H}_1$ fall under the scope of Theorems 2 and 4 of [**3**]. The proof given in [**3**] of Theorem 4 uses complex interpolation and it is quite technical. Ours involves only elementary estimates; moreover, we are able to control $\tilde{H}_1$ from above by proving (1). This is most important for the mentioned application and it is not proved in [**3**].

The paper is structured as follows. In §1 we are concerned with the one-dimensional case and with the maximal Carleson operator. In §§2 and 3, respectively, we study $H_1, \tilde{H}_1$ and $H_2, \tilde{H}_2$. In §4 we consider an even more general operator $\tilde{H}_3$ (where the Sup is taken over all regions). We give a counterexample to show that $\tilde{H}_3$ is not a bounded operator. This sets a halt to our generalisations of the maximal double Hilbert transform. Finaly, in §5 we say some more about the application we mentioned.

By c we denote a constant not necessarily the same in all instances.

1. The one-dimensional case. There exists a C^∞ function $\phi(x')$ supported on $\{|x'| \le 2\pi\}$ such that if we write $\phi_k(x') = 2^k\phi(2^k x')$, then $1/x' = \sum_{k=0}^\infty \phi_k(x')$ for $|x'| \le \pi$. Let J be a fixed subset of the nonnegative integers N and let us consider the operators

$$Hf(x,y) = \int \sum_{k\in J} \phi_k(x') f(x-x')\,dx'$$

and

$$\tilde{H}f(x) = \operatorname*{Sup}_{K>0} \left| \int \sum_{\substack{k\in J\\ k\le K}} \phi_k(x') f(x-x')\,dx' \right|.$$

$\tilde{H}$ is clearly a variant of the maximal Hilbert transform. We have

LEMMA 1. *H and $\tilde{H}$ are bounded operators on $L_p(T)$, $1 < p < \infty$, with norm independent of J. Moreover, the following inequality holds:*

$$\tilde{H}f(x) \leq c\{Mf(x) + M(Hf)(x)\}. \tag{3}$$

REMARK. This is exactly Lemma 3 of [**2**]. We are going to prove it for the reader's convenience and for an inaccuracy that appears in [**2**]. Namely, one needs to use a smooth cut-off function like the following $\theta(x')$ rather than a sharp one.

PROOF. Clearly, ϕ has the following properties:

1. $\hat{\phi}(0) = 0$,
2. $|\hat{\phi}(\xi)| \leq c_M/|\xi|^M$ for $|\xi| > 1$ and for any integer $M \geq 0$,
3. $|\hat{\phi}(\xi)| \leq c|\xi|$ for $|\xi| \leq 1$.

Since $\hat{\phi}_k(\xi)$ is mainly supported on $|\xi| \sim 2^k$, we have that

$$|m_K(\xi)| = \left|\sum_{\substack{k\in J\\ k\leq K}} \hat{\phi}_k(\xi)\right| \leq \sum_{k=0}^{\infty} |\hat{\phi}_k(\xi)| \leq c \tag{a}$$

independently of J and K. We have $m'_K(\xi) = \sum_{k\in J, k\leq K} \hat{\psi}_k(\xi)$, where $\psi(x') = x'\phi(x')$ and $\psi_k(x') = 2^k x'\phi(2^k x') = \psi(2^k x')$. Now $\psi(x')$ is C^∞, compactly supported and, moreover,

4. $|\hat{\psi}(\xi)| \leq c$ for every ξ,
5. $|\hat{\psi}(\xi)| \leq c_M/|\xi|^M$ for $|\xi| > 1$ and for any integer $M \geq 0$.

Therefore, for any dyadic interval $I \subset \mathbf{R}$ we have that

$$\int_I |m'_K(\xi)|\, d\xi \leq \int_I \sum_{k=0}^{\infty} |\hat{\psi}_k(\xi)|\, d\xi \leq c. \tag{b}$$

By the Marcinkiewicz multiplier theorem, the operator

$$H_K f(x) = \int \sum_{\substack{k\in J\\ k\leq K}} \phi_k(x') f(x - x')\, dx'$$

is bounded on L_p with norm independent of K. By a standard argument there exists $\lim_{K\to\infty} H_K f(x) = Hf(x)$ in L_p norm and $\|Hf\|_p \leq c_p \|f\|_p$, $1 < p < \infty$.

Now let $\theta(x')$ be a positive C^∞ function supported on $\{|x'| \leq 1\}$ and such that $\int_{-1}^{1} \theta(x')\, dx' = 1$. To prove equation (3) we are going to show that

$$\left|\sum_{\substack{k\in J\\ k\leq K}} \phi_k(x') - \theta_K * \sum_{k\in J} \phi_k(x')\right| \leq c2^{-K}/(x')^2 + 2^{-2K},$$

where $\theta_K(x') = 2^K \theta(2^K x')$. If $|x'| < 100 2^K$ then $\|\sum_{k\in J, k\leq K} \phi_k(x')\|_\infty \leq c2^K$ and

$$\left\|\theta_K * \sum_{k\in J} \phi_k(x')\right\|_\infty \leq \left\|\check{\theta}_K \cdot \sum_{k\in J} \check{\phi}_k(\xi)\right\|_1 \leq c\|\check{\theta}_K\|_1 \leq c2^K.$$

If, instead, $|x'| \geq 1002^K$, then $\sum_{k\in J}\phi_k(x') = \sum_{k\in J, k\leq K}\phi_k(x')$, and so for a suitable $\tilde{x} = \tilde{x}(x'')$ we have that

$$\left|\sum_{\substack{k\in J\\ k\leq K}}\phi_k(x') - \theta_K * \sum_{k\in J}\phi_k(x')\right| = \left|\int \sum_{\substack{k\in J\\ k\leq K}}(\phi_k(x') - \phi_k(x'-x''))\theta_K(x'')\,dx''\right|$$
$$\leq c\int \sum_{\substack{k\in J\\ k\leq K}}|\phi_k'(\tilde{x})|2^{-K}\theta_K(x'')\,dx'' \leq c2^{-K}/(x')^2.$$

Hence the claim is proved. Now if we write $P_K(x') = 2^{-K}/(x')^2 + 2^{-2K}$ we have that

$$\left|\sum_{\substack{k\in J\\ k\leq K}}\phi_k * f(x)\right| \leq c\left\{\operatorname*{Sup}_K\left|\theta_K * \sum_{k\in J}\phi_k * f(x)\right| + \operatorname*{Sup}_K|P_K * f(x)|\right\}$$
$$\leq c\{M(Hf)(x) + Mf(x)\}.$$

Therefore the lemma is proved.

The following are called Carleson operator and Carleson maximal operator:

$$Cf(x) = \int_{-\pi}^{\pi}\exp(iN(x)x')/x'f(x-x')\,dx',$$
$$\tilde{C}f(x) = \operatorname*{Sup}_{\varepsilon>0}\left|\int_{\varepsilon<|x'|\leq\pi}\exp(iN(x)x')/x'f(x-x')\,dx'\right|,$$

where $N(x)$ is any measurable bounded integer valued function. We have

PROPOSITION 1. *The operators C and $\tilde{C}$ are bounded from $L_p(T)$ to itself, $1 < p < \infty$, with norm independent of $N(x)$. Moreover, the following inequality holds: $\tilde{C}f(x) \leq c\{Mf(x) + M(Cf)(x)\}$.*

PROOF. $Cf(x)$ is pointwise dominated by the maximal partial sums operator $\operatorname{Sup}_n|\int_{|x'|\leq\pi}\exp(inx')/x'f(x-x')\,dx'|$ (also called Carleson operator) whose boundedness has been proved in [**4**]. As for $\tilde{C}f(x)$ one might go through Carleson and Hunt's proof and see that it shows that $\tilde{C}$ is also bounded, or observe that (see [**6**, p. 218])

$$\tilde{C}f(x) \leq \operatorname*{Sup}_{\varepsilon>0, n\in Z}\left|\int_{\varepsilon<|x'|\leq\pi}e^{inx'}/x'f(x-x')\,dx'\right|$$
$$\leq c\left\{\operatorname*{Sup}_n\left\{M(e^{inx'}f(x'))(x) + M\left(\left|\int_{|x''|\leq\pi}e^{inx''}/x''f(x'-x'')\,dx''\right|\right)(x)\right\}\right\}$$
$$\leq c\{Mf(x) + M(Cf)(x)\}.$$

This proves the desired inequality and concludes the proof.

2. The first variant: $\tilde{H}_1$. Let $B \subset N \times N$. We consider the operator

$$H_1 f(x,y) = \iint_{(k,h)\in B} \phi_k(x')\phi_h(y')f(x-x',y-y')\,dx'\,dy'.$$

We have the following

THEOREM 1. *H_1 is a bounded operator on $L_p(T^2)$, $1 < p < \infty$, with norm independent of B.*

PROOF. Since $\hat{\phi}_k(\xi)\hat{\phi}_h(\eta)$ is mainly supported on $\{|\xi| \sim 2^k,\ |\eta| \sim 2^h\}$, as in Lemma 1, one can show by the Marcinkiewicz multiplier theorem that the operator

$$H^1_{k_0h_0} f(x,y) = \iint \sum_{\substack{(k,h)\in B \\ k\le k_0,h\le h_0}} \phi_k(x')\phi_h(y')f(x-x',y-y')\,dx'\,dy'$$

is bounded on L_p with norm independent of k_0, h_0 and B. By a standard argument one can show that there exists $\lim_{k_0\to\infty,h_0\to\infty} H^1_{k_0h_0}f(x,y) = H_1 f(x,y)$ in L_p norm by checking it on a dense subset of functions f. It suffices to consider $f(x',y') = f_1(x')f_2(y')$, f_i smooth. Then H_1 is bounded on L_p.

Now we have

THEOREM 2. *Let J_1 and J_2 be subsets of N. Let $B \subset J_1 \times J_2$ be a collection of pairs (k,h) such that:*

(i) *For every k the section $B_k = \{h \in J_2\colon\ (k,h) \in B\}$ is a truncation of J_2 possibly depending upon k.*

(ii) *For every h the section $B_h = \{k \in J_1\colon\ (k,h) \in B\}$ is a truncation of J_1 possibly depending upon h.*

Then the operator

$$\tilde{H}_1 f(x,y) = \operatorname*{Sup}_{k_0,h_0} \left| \iint \sum_{\substack{(k,h)\in B \\ k\le k_0,h\le h_0}} \phi_k(x')\phi_h(y')f(x-x',y-y')\,dx'\,dy' \right|$$

is bounded from $L_p(T^2)$, $1 < p < \infty$, to itself with norm independent of B. Moreover, if $\tilde{H}$ is defined as in the preceding section the following inequality holds:

$$\tilde{H}_1 f(x,y) \le c\{M_{x'}M_{y'}f(x,y) + M_{x'}\tilde{H}_{y'}f(x,y) + M_{y'}\tilde{H}_{x'}f(x,y) + M_{x'}M_{y'}(H_1 f)(x,y)\}. \tag{4}$$

PROOF. We consider the convolution kernel

$$G_{k_0h_0}(x',y') = \sum_{\substack{(k,h)\in B \\ k\le k_0,h\le h_0}} \phi_k(x')\phi_h(y') - \theta_{k_0}(x')\theta_{h_0}(y') * \sum_{(k,h)\in B} \phi_k(x')\phi_h(y').$$

We claim that $|G_{k_0h_0} * f(x,y)|$ is dominated by the right-hand side of (4). To prove it we shall subdivide T^2 into four regions R_i, $i = 1,\dots,4$, and define $G^i_{k_0h_0}(x',y') = G_{k_0h_0}(x',y')\chi_{R_i}(x',y')$. The first region is defined as $R_1 = \{|x'| \le 1002^{-k_0}, |y'| \le 1002^{-h_0}\}$. Now $|G^1_{k_0h_0}(x',y')| \le c2^{k_0}2^{h_0}$ as in the proof of Lemma 1. Hence

$|G^1_{k_0h_0} * f(x,y)| \le cM_{x'}M_{y'}f(x,y)$. The second region is defined as $R_2 = \{|x'| < 1002^{-k_0}, |y'| > 1002^{-h_0}\}$. We have that

$$\begin{aligned}|G^2_{k_0h_0} * f(x,y)| &\le c2^{k_0}\chi_{\{|x'|\le 1002^{-k_0}\}}(x') \sum_{k=k_0-1g200}^{k_0} \left| \sum_{\substack{h\in B_k\\ h\le h_0}} \phi_h(y') * f(x,y)\right| \\ &\quad + c2^{k_0}2^{h_0}\chi_{\{|x'|\le 1002^{-k_0}\}}(x')\chi_{\{|y'|\le 1002^{-h_0}\}}(y') \\ &\quad * \left|\sum_{(k,h)\in B} \phi_k(x')\phi_h(y') * f(x,y)\right| \\ &\le c\{M_{x'}\tilde{H}_{y'}f(x,y) + M_{x'}M_{y'}(H_1f)(x,y)\}.\end{aligned}$$

Similarly we define $R_3 = \{|x'| > 1002^{-k_0}, |y'| < 1002^{-h_0}\}$ and we obtain

$$|G^3_{k_0h_0} * f(x,y)| \le c\{M_{y'}\tilde{H}_{x'}f(x,y) + M_{x'}M_{y'}(H_1f)(x,y)\}.$$

We are left with $R_4 = \{|x'| \ge 1002^{-k_0}, |y'| \ge 1002^{-h_0}\}$. Now

$$\begin{aligned}|G^4_{k_0h_0}(x',y')| &= \left|\theta_{k_0}\theta_{h_0} * \sum_{\substack{(k,h)\in B\\ k\le k_0, h\le h_0}} \phi_k(x')\phi_h(y') - \sum_{\substack{(k,h)\in B\\ k\le k_0, h\le h_0}} \phi_k(x')\phi_h(y')\right| \\ &= \left|\iint \left\{ \sum_{\substack{(k,h)\in B\\ k\le k_0, h\le h_0}} \phi_k(x'-x'')\phi_h(y'-y'') - \phi_k(x')\phi_h(y')\right\}\right. \\ &\qquad \left. \times\ \theta_{k_0}(x'')\theta_{h_0}(y'')\, dx''\, dy''\right| \\ &\le \left|\iint \left\{ \sum_{\substack{(k,h)\in B\\ k\le k_0, h\le h_0}} \phi_k(x'-x'')\{\phi_h(y'-y'') - \phi_h(y')\}\right\}\right. \\ &\qquad \left. \times\ \theta_{k_0}(x'')\theta_{h_0}(y'')\, dx''\, dy''\right| \\ &\quad + \left|\iint \sum_{\substack{(k,h)\in B\\ k\le k_0, h\le h_0}} \phi_h(y')\{\phi_k(x'-x'') - \phi_k(x')\}\theta_{k_0}(x'')\theta_{h_0}(y'')\, dx''\, dy''\right| \\ &= \overline{G}^4_{k_0h_0}(x',y') + \overline{\overline{G}}^4_{k_0h_0}(x',y').\end{aligned}$$

We observe that

$$|\overline{\overline{G}}^4_{k_0h_0}(x',y')| = \left| \int \sum_{k\le k_0} \{\phi_k(x'-x'') - \phi_k(x')\}\theta_{k_0}(x'') \sum_{\substack{h\in B_k\\ h\le h_0}} \phi_h(y')\,dx'' \right|.$$

Hence

$$\begin{aligned} |\overline{\overline{G}}^4_{k_0h_0} * f(x,y)| &\le \int \sum_{k\le k_0} \left(\int |\phi_k(x'-x'') - \phi_k(x')|\theta_{k_0}(x'')\,dx'' \right) \\ &\qquad \cdot \operatorname*{Sup}_{h_0} \left| \int \sum_{\substack{h\in B_k\\ h\le h_0}} \phi_h(y') f(x-x', y-y')\,dy' \right| dx' \\ &\le c \int P_{k_0}(x') |\tilde{H}_{y'} f(x-x',y)|\,dx' \le cM_{x'}\tilde{H}_{y'}f(x,y). \end{aligned}$$

Now we write

$$\begin{aligned} \overline{G}^4_{k_0h_0}(x',y') &= \iint \sum_{\substack{(k,h)\in B\\ k\le k_0, h\le h_0}} \{\phi_h(y'-y'') - \phi_h(y')\}\{\phi_k(x'-x'') - \phi_k(x')\} \\ &\qquad \cdot \theta_{k_0}(x'')\theta_{h_0}(y'')\,dx''\,dy'' \\ &\quad + \iint \sum_{\substack{(k,h)\in B\\ k\le k_0, h\le h_0}} \{\phi_h(y'-y'') - \phi_h(y')\}\phi_k(x')\theta_{k_0}(x'')\theta_{h_0}(y'')\,dx''\,dy'' \\ &= \tilde{G}^4_{k_0h_0}(x',y') + \tilde{\tilde{G}}^4_{k_0h_0}(x',y'). \end{aligned}$$

We start by studying

$$\begin{aligned} |\tilde{G}^4_{k_0h_0}(x',y')| &\le \iint \sum_{h\le h_0} |\phi_h(y'-y'') - \phi_h(y')|\theta_{h_0}(y'') \\ &\qquad \cdot \sum_{k\le k_0} |\phi_k(x'-x'') - \phi_k(x')|\theta_{k_0}(x'')\,dx''\,dy''. \end{aligned}$$

Therefore, $|\tilde{G}^4_{k_0h_0} * f(x,y)| \le cP_{h_0}(y') \cdot P_{k_0}(x') * f(x,y) \le cM_{x'}M_{y'}f(x,y)$.

We are left to study the action of

$$\tilde{\tilde{G}}^4_{k_0h_0}(x',y') = \int \sum_{h\le h_0} \{\phi_h(y'-y'') - \phi_h(y')\}\theta_{h_0}(y'') \sum_{\substack{k\in B_h\\ k\le k_0}} \phi_k(x')\,dy''.$$

Now

$$|\tilde{\tilde{G}}^4_{k_0h_0} * f(x,y)| \le cP_{h_0}(y') \operatorname*{Sup}_{k_0} \left| \sum_{\substack{k\in B_h\\ k\le k_0}} \phi_k(x') * f \right| (x,y) \le cM_{y'}\tilde{H}_{x'}f(x,y).$$

So the claim has been proved. Now observe that

$$\left| \iint \sum_{\substack{(k,h)\in B \\ k\le k_0, h\le h_0}} \phi_k(x')\phi_h(y')f(x-x',y-y')\,dx'\,dy' \right|$$

$$\le |G_{k_0h_0} * f(x,y)| + \left| \theta_{k_0}(x')\theta_{h_0}(y') * \sum_{(k,h)\in B} \phi_k(x')\phi_h(y') * f(x,y) \right| .$$

The last term is dominated by $cM_{x'}M_{y'}(H_1f)(x,y)$. This ends the proof.

3. The second variant: $\tilde{H}_2$. For every x let B_x be a fixed subset of $N\times N$ with the following property:

(ii′) for every h there exists an integer $r(x,h)$ such that $B_{xh} = \{k\in N\colon (k,h)\in B_x\} = \{k \ge r(x,h)\}$.

We consider the operator

$$H_2f(x,y) = \iint \sum_{(k,h)\in B_x} \phi_h(y')\phi_k(x')f(x-x',y-y')\,dx'\,dy'.$$

The following theorem holds.

THEOREM 3. *In the assumption* (ii′) *there exists a constant c_p depending only upon p such that $\|H_2f\|_p \le c_p\|f\|_p$, $1<p<\infty$.*

PROOF. We are going to prove that the operator

$$H^2_{k_0h_0}f(x,y) = \iint \sum_{\substack{(k,h)\in B_x \\ k\le k_0, h\le h_0}} \phi_h(y')\phi_k(x')f(x-x',y-y')\,dx'\,dy'$$

that for simplicity we also denote by $H_2f(x,y)$, is bounded on L_p with bound depending only upon p. For this purpose it is enough to consider those f which are smooth. The proof is based on (a) and (b) of §1. Let us denote by S the classical S-function acting on functions of one variable, the variable y in our case. By the Littlewood-Paley theorem we know that for a.e. x fixed

$$\left\| \left(\sum_J |S_J H_2 f(x,y)|^2 \right)^{1/2} \right\|_{L_p(dy)} \sim \|H_2f(x,y)\|_{L_p(dy)}.$$

We raise this relation to the pth power and integrate it with respect to x, obtaining

$$\left\| \left(\sum_J |S_J H_2 f(x,y)|^2 \right)^{1/2} \right\|^p_{L_p(dydx)} \sim \|H_2f(x,y)\|^p_{L_p(dydx)}.$$

Therefore to prove the boundedness of H_2 it suffices to prove that

$$\left\| \left(\sum_J S_J H_2 f(x,y)|^2 \right)^{1/2} \right\|_{L_p(dydx)} \le c_p\|f\|_p. \tag{5}$$

Let η denote the dual variable of y. In estimating the left-hand side of (5) we consider the dyadic intervals J of each half line separately. Assume from now on that J belongs to $\{\eta \geq 0\}$. For a.e. x fixed we have that

$$\begin{aligned} S_J H_2 f(x,y) &= \int_J e^{i\eta y} \sum_{h \leq h_0} \hat{\phi}_h(\eta) \sum_{k \geq r(x,h)} \phi_k(x') * f(x,y)(\eta)\, dn \\ &= \int_J e^{i\eta y} \sum_{h \leq h_0} \left\{ \int_{2^{\bar{h}}}^{\eta} (\hat{\phi}_h)'(t)\, dt + \hat{\phi}_h(2^{\bar{h}}) \right\} \sum_{k \geq r(x,h)} \phi_k(x') * \hat{f}(x,\eta)\, d\eta \\ &= S_J^{(1)} H_2 f(x,y) + S_J^{(2)} H_2 f(x,y). \end{aligned}$$

Let $J = [2^{\bar{h}}, 2^{\bar{h}+1})$, $h \gtreqless 0$. By exchanging the order of integration we have

$$(6)\quad |S_J^{(2)} H_2 f(x,y)| \leq \sum_{h \leq h_0} |\hat{\phi}_h(2^{\bar{h}})| \cdot \left| \sum_{k \geq r(x,h)} \phi_k(x') * S_J f(x,y) \right| \leq c\tilde{H} S_J f(x,y).$$

Next we turn to

$$S_J^{(1)} H_2 f(x,y) = \int_J \sum_{h \leq h_0} (\hat{\phi}_h)'(t) \left\{ \int_t^{2^{\bar{h}+1}} e^{i\eta y} \sum_{k \geq r(x,h)} \phi_k(x') * \hat{f}(x,\eta)\, d\eta \right\} dt.$$

Now for every $t \in J$ let us denote by S_t the multiplier transformation corresponding to the interval $[t, 2^{\bar{h}+1})$. Using the relation $S_t S_J = S_t$, Schwarz inequality and (b) we obtain that

$$\begin{aligned} |S_J^{(1)} H_2 f(x,y)| &\leq \int_J \sum_h |(\hat{\phi}_h)'(t)| \cdot \left| \sum_{k \geq r(x,h)} \phi_k(x') * S_t S_J f(x,y) \right| dt \\ &\leq c \left(\int_J \sum_h |(\hat{\phi}_h)'(t)| \cdot |\tilde{H} S_t S_J f(x,y)|^2\, dt \right)^{1/2}. \end{aligned}$$

If we write $d\gamma(t) = \sum_h |\hat{\phi}_h(t)| dt$ and we use Theorem 4″ on p. 103 of [**7**], the Littlewood-Paley theorem, Theorem 1 of [**9**] and Proposition 2 of [**10**] we obtain

$$(7)\quad \begin{aligned} &\left\| \left(\sum_J |S_J^{(1)} H_2 f(x,y)|^2 \right)^{1/2} \right\|_{L_p(dy\,dx)}^p \\ &\leq c \left\| \left(\int_0^{+\infty} |\tilde{H}_{x'} S_t S_J f(x,y)|^2\, d\gamma(t) \right)^{1/2} \right\|_{L_p(dy\,dx)}^p \\ &\leq c_p \left\| \left(\sum_J |S_J f(x',y)|^2 \right)^{1/2} \right\|_{L_p(dy\,dx)}^p \leq c_p \|f_2\|_p^p. \end{aligned}$$

Now we observe that

$$\left\| \left(\sum_J |S_J H_2 f(x,y)|^2 \right)^{1/2} \right\|^p_{L_p(dy\,dx)}$$
$$\leq \left\| \left(\sum_J |S_J^{(1)} H_2 f(x,y)|^2 \right)^{1/2} + \left(\sum_J |S_J^{(2)} H_2 f(x,y)|^2 \right)^{1/2} \right\|^p_{L_p(dy\,dx)}.$$

Now if we use (6) and (7) we obtain (5). So we proved that the operators $H^2_{k_0 h_0}$ are uniformly bounded on L_p, $1 < p < \infty$. Then to show that there exists

$$\lim_{k_0, h_0 \to \infty} H^2_{k_0 h_0} f(x,y) = H_2 f(x,y)$$

in L_p norm it is enough to apply $H^2_{k_0 h_0}$ to $f(x', y') = f_1(x') f_2(y')$ where f_i are smooth. Finally one can prove that H_2 is bounded on L_p.

Then we consider the operator

$$\tilde{H}_2 f(x,y) = \operatorname*{Sup}_{h_0} \left| \iint \sum_{\substack{(k,h) \in B_x \\ h \leq h_0}} \phi_h(y') \phi_k(x') f(x - x', y - y')\, dx\, dy' \right|.$$

We have

THEOREM 4. *If* (ii′) *is satisfied then there exists a constant c_p depending only upon p such that $\|\tilde{H}_2 f\|_p \leq c_p \|f\|_p$, $1 < p < \infty$. Moreover, the following inequality holds:*

$$\tilde{H}_2 f(x,y) \leq c\{ M_{y'} \tilde{H}_{x'} f(x,y) + M_{\overline{y}}(H_2 f(x, \overline{y}))(y) \}.$$

PROOF. We observe that in the formula defining $H_2 f(x,y)$, once the convolution on the x' variable has been performed, the operator acting on the y' variable is a constant coefficients singular integral. Hence proceeding as in the proof of Theorem 2 one can show the stated inequality. As a consequence $\tilde{H}_2$ is bounded on L_p.

4. A counterexample. It would be useful for the applications to be able to use the operator

$$\tilde{H}_3 f(x,y) = \operatorname*{Sup}_{B} \left| \iint \sum_{(k,h) \in B} \phi_h(y') \phi_k(x') f(x - x', y - y')\, dx'\, dy' \right|,$$

with B an arbitrary subset of $N \times N$. Unfortunately such an operator is unbounded even on L_2 as it is shown by the following counterexample which has been pointed out to us by C. Fefferman. Consider the operator

$$Kf(x) = \sum_{k \in B_x} \phi_k * f(x).$$

We are going to show how to define B_x such that $Kf(x) = \infty$ at every x for a suitable f belonging to $L_2(T)$. Let $f \sim \sum_{n=1}^{\infty} \{\exp(i\lambda_n x)\}/n$, where the λ_n's are sparse, for instance $\lambda_n = 2^{2^n}$. Observe that $\phi_{\lambda_n} * f(x) = c_0 \{\exp(i\lambda_n x)\}/n + o(2^{-n})$, where $c_0 = \hat{\phi}_{\lambda_n}(\lambda_n) = \hat{\phi}(1) \neq 0$ as we may assume. Let us subdivide $\{\theta\colon\ 0 < \theta \leq 2\pi\}$ into four disjoint quadrants $Q_1, \dots, Q_4$, where $Q_1 = \{\theta\colon\ -\pi/4 < \theta \leq \pi/4\}$. For every x fixed there exists at least one of the Q_i's say Q_1, which contains infinitely many $\lambda_n x$, $n = 1, 2, \dots$, and such that $\sum_{\lambda_n x \in Q_1} 1/n = \infty$. Define $B_x = \{k = \lambda_n\colon\ \lambda_n x \in Q_1\}$. We have that $|\mathrm{Re}(\exp(ikx))|$ or $|\mathrm{Im}(\exp(ikx))| > c$ for every $k \in B_x$. So $|\sum_{k \in B_x} \phi_k * f(x)| = \infty$.

5. An application. These results have an application to the study of the operator

$$Tf(x,y) = \iint \exp\{i(N(x,y)x' + N^2(x,y)y')\}/x'y' f(x-x', y-y')\, dx'\, dy',$$

where $N(x,y)$ is a measurable bounded integer valued function. $Tf(x,y)$ is the maximal partial sums operator $\mathrm{Sup}_N |S_{N,N^2} f(x,y)|$, where

$$S_{N,N^2} f(x,y) = \sum_{\substack{|n| \leq N \\ |m| \leq N^2}} a_{nm} \cdot \exp\{i(nx + my)\}.$$

To prove a.e. convergence of $S_{N,N^2} f(x,y)$ one looks for an estimate of $Tf(x,y)$ which does not depend upon $N(x,y)$. In [**5**] we proved a uniform estimate in L_p of $Tf(x,y)$ for $N(x,y) = [\lambda xy]$, $\lambda > 10^{10}$, using Theorems 1 and 2. Moreover, we gave an example of a family of functions $N(x,y)$ which cannot be handled by Theorems 1 and 2 and which leads instead to H_2 and $\tilde{H}_2$. Let us finally observe that the boundedness of $\tilde{H}_1$ and $\tilde{H}_2$ is enough for the estimate of pairs of norm 1 while (1) and (2) are needed for the estimate of pairs of norm smaller than 1.

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